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Scattering by a toroidal coil

Ph Roux[1]

Department of Mathematics, University of Rennes, Campus Beaulieu, 35042, Rennes, France

E-mail: Philippe.Roux@univ-rennes1.fr

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Abstract
In this paper we consider the Schrödinger operator in $\mathbb{R}^3$ with a long-range magnetic potential associated with a magnetic field supported inside a torus $\mathbb{T}$. Using the scheme of smooth perturbations we construct stationary modified wave operators and the corresponding scattering matrix $S(\lambda)$. We prove that the essential spectrum of $S(\lambda)$ is an interval of the unit circle depending only on the magnetic flux ϕ across the section of $\mathbb{T}$. Additionally we show that, in contrast to the Aharonov–Bohm potential in $\mathbb{R}^2$, the total scattering cross-section is always finite. We also conjecture that the case treated here is a typical example in dimension 3.

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1. Introduction

Let $A(x)$ be a magnetic potential

$$A(x) = \frac{a(\varphi_x)}{|x|}\,\mathbf{e}_{\varphi_x} \qquad |x| \geqslant R > 0 \tag{1.1}$$

where $a \in C_0^\infty(0,\pi)$ is a positive function of the colatitude of $x = (x_1, x_2, x_3)$

$$\varphi_x = \arccos\left(\frac{x_3}{|x|}\right) \in [0,\pi] \tag{1.2}$$

and $\mathbf{e}_{\varphi_x}$ denotes the basis vector of spherical coordinates $\left(\mathbf{e}_{r_x}, \mathbf{e}_{\varphi_x}, \mathbf{e}_{\theta_x}\right)$ associated with the point x

$$\mathbf{e}_{\varphi_x} = \frac{1}{|x|}\left(\frac{x_1x_3}{\sqrt{x_1^2+x_2^2}}, \frac{x_2x_3}{\sqrt{x_1^2+x_2^2}}, -\sqrt{x_1^2+x_2^2}\right). \tag{1.3}$$

[1] Permanent address: IUT Lannion, Département Informatique, rue Branly BP150, 22302 Lannion Cedex, France.

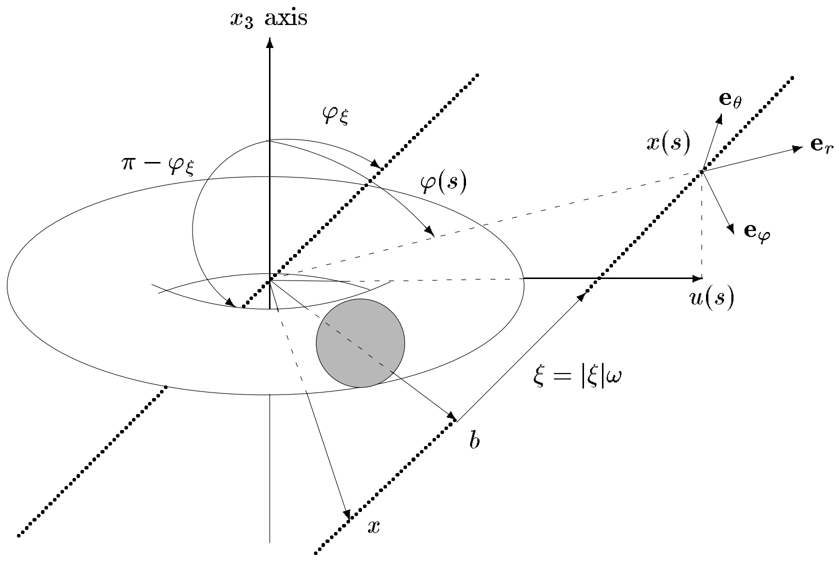

Figure 1. The toroidal coil $\mathbb{T}$.

Physically the potential (1.1) corresponds to a magnetic field $B = \operatorname{curl} A$ supported inside a torus $\mathbb{T}$ obtained by revolution around the x_3 axis (see figure 1). The function a in (1.1) depends only on the section of $\mathbb{T}$ and the flux ϕ of B across any section of the torus

$$\phi = \int_0^\pi a(\varphi)\,\mathrm{d}\varphi > 0. \tag{1.4}$$

This situation is known, from the work of Aharonov and Bohm [AB59], to show a purely quantum phenomenon: a compactly supported magnetic field can act on particles which never cross its support. From the mathematical point of view, despite B as a finite support, the potential A decays as $|x|^{-1}$ at infinity and is of long-range nature. Thus one expects the properties of the scattering process associated with the potential (1.1) to be different from the case of short-range potentials.

Here we consider the Schrödinger operator

$$H = (D - A(x))^2 \qquad D = -\mathrm{i}\nabla_x \tag{1.5}$$

in $\mathbb{L}^2(\mathbb{R}^3)$, and develop the scheme of smooth perturbations for the pair $H, H_0 = -\Delta$. Although the usual wave operators exist (due to the transversal gauge condition $\langle A(x), x\rangle = 0$, for all $x \in \mathbb{R}^3$, see [LT87]) we prefer to work with modified wave operators of the Isozaki–Kitada type:

$$W_\pm(H, H_0, J) = s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} J\, \mathrm{e}^{-\mathrm{i}tH_0} \tag{1.6}$$

with stationary identifications $J = J_\pm$ depending on the sign of t, as in [Nic94, RY02b]. We choose the operators $J_\pm$ as pseudo-differential operators (PDO) with symbols $\exp(\mathrm{i}\Phi_\pm(x,\xi))$ such that the effective perturbation $T_\pm = HJ_\pm - J_\pm H_0$ is short-range, that is the phase function $\Phi_\pm$ satisfies $\nabla_x \Phi_\pm(x,\xi) = A(x)$. Thus the existence of wave operators (1.6) relies only on the *limiting absorption principle* in contrast to [RY02b] where the *radiation estimate* was needed. Since the identifications $J_\pm$ are 'close' to unitary operators, the wave operators $W_\pm(H, H_0, J_\pm)$ are automatically isometric and complete, indeed, they coincide with the usual wave operators $W_\pm(H, H_0) = W_\pm(H, H_0, Id)$.

The scattering operator, defined by $S = W_+^*(H, H_0)W_-(H, H_0)$, commutes with H_0; so, in the spectral representation of H_0, it reduces to the multiplication by the operator-valued function $S(\lambda)$, called the scattering matrix (SM) which acts as an integral operator on the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$. Our study of the SM relies on its stationary representation

$$S(\lambda) = \mathcal{W}(\lambda) - 2\mathrm{i}\pi\,\Gamma_0(\lambda)(J_+^*T_- - T_+^*R(\lambda + \mathrm{i}0)T_-)\Gamma_0^*(\lambda) \tag{1.7}$$

where $R(z) = (H - z)^{-1}$, and

$$\mathcal{W}(\lambda) = \Gamma_0(\lambda)W_+(H_0, H_0, J_+^*J_-)\Gamma_0^*(\lambda) \tag{1.8}$$

with $\Gamma_0(\lambda) : \mathbb{L}^2(\mathbb{R}^3) \longrightarrow \mathbb{L}^2(\mathbb{S}^2)$ defined for u in the Schwarz class by

$$(\Gamma_0(\lambda)u)(\omega) = \frac{\sqrt{\lambda}}{2(2\pi)^{3/2}}\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\sqrt{\lambda}\langle\omega,x\rangle}u(x)\,\mathrm{d}x \qquad \omega \in \mathbb{S}^2 \tag{1.9}$$

and $\Gamma_0^*(\lambda)$ is formally adjoint to $\Gamma_0(\lambda)$. To justify the formula (1.7) we decompose it as a sum of bounded operators. First, we calculate the term $\mathcal{W}(\lambda)$ and prove that it reduces to the operator of multiplication by the function w defined on $\mathbb{S}^2$ by

$$\mathrm{w}(\omega) = \exp\left(\mathrm{i}\int_{\pi-\varphi_\omega}^{\varphi_\omega} a(\varphi)\,\mathrm{d}\varphi\right) \qquad \omega \in \mathbb{S}^2. \tag{1.10}$$

Then we show that the remaining term, $S(\lambda) - \mathcal{W}(\lambda)$, is an integral operator on $\mathbb{S}^2$ with a C^∞ kernel. Thus, we can make a spectral analysis of the SM. Since $S(\lambda)$ is a compact perturbation of $\mathcal{W}(\lambda)$, we calculate its essential spectrum, that is

$$\sigma_{\mathrm{ess}}(S(\lambda)) = \{\mu = \exp(\mathrm{i}\nu) \in \mathbb{C} \mid \nu \in [-\phi, \phi]\}\,. \tag{1.11}$$

In particular $\sigma_{\mathrm{ess}}(S(\lambda))$ depends only on the magnetic flux ϕ (1.4) of B across the section of $\mathbb{T}$. Now if we take as a definition of the differential scattering cross-section

$$\Sigma_{\mathrm{diff}}(\omega, \omega_0; \lambda) = \frac{\lambda^{-(d-1)/2}}{(2\pi)^{d-1}}|\mathrm{s}(\omega, \omega_0; \lambda)|^2 \qquad \omega \neq \omega_0 \tag{1.12}$$

with $d = 3$ and where ω_0 (resp. ω) is the incoming (outgoing) direction, then the function $\Sigma_{\mathrm{diff}}(\omega, \omega_0; \lambda)$ belongs to $C^\infty(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}^+)$. In particular the total scattering cross-section

$$\Sigma_{\mathrm{tot}}(\omega_0; \lambda) = \int_{\mathbb{S}^2} \Sigma_{\mathrm{diff}}(\omega, \omega_0; \lambda)\,\mathrm{d}\omega \tag{1.13}$$

is finite for all incident directions $\omega_0 \in \mathbb{S}^2$.

This paper is organized as follows: in section 2 we construct stationary wave operators and recover the basic results of scattering theory for potential (1.1); in section 3 we analyse the structure of the SM and its spectral properties; finally, in section 4 we make some remarks about this example and the two-dimensional Aharonov–Bohm effect, we also conjecture that the situation described here is very general in the three-dimensional case.

2. Wave operators

In this section we construct time-independent modified wave operators, as in [Nic94, RY02b], and recover basic results on long-range magnetic scattering in the transversal gauge [LT87].

2.1. Construction of identifications

In the scheme of smooth perturbations the choice of identifications $J = J_\pm$ in (1.6) is determined by the condition that the effective perturbation $T_\pm = HJ_\pm - J_\pm H_0$ be 'short-range'. If, as in [Yaf98], we search $J_\pm$ as a PDO with symbol $j_\pm(x,\xi)$ then the function $\Psi_\pm(x,\xi) = \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle} j_\pm(x,\xi)$ should be an approximate (i.e. up to short-range terms) eigenfunction of H associated with the eigenvalue $|\xi|^2$. Thus we set $j_\pm(x,\xi) = \exp(\mathrm{i}\Phi_\pm(x,\xi))$ and compute

$$(H - |\xi|^2)\Psi_\pm(x,\xi) = (2\langle \xi, \nabla_x\Phi_\pm(x,\xi) - A(x)\rangle + |\nabla_x\Phi_\pm(x,\xi) - A(x)|^2 \\ - \mathrm{i}\,\mathrm{div}_x(\nabla_x\Phi_\pm(x,\xi) - A(x)))\Psi_\pm(x,\xi). \tag{2.1}$$

Taking only the principal (i.e. the first) term of (2.1), we obtain the eikonal equation for $\Phi_\pm$

$$\langle \xi, \nabla_x\Phi_\pm(x,\xi) - A(x)\rangle = 0. \tag{2.2}$$

As shown in [Yaf98], this equation admits solutions with decaying derivatives for large $|x|$

$$\Phi_\pm(x,\xi) = \mp\int_0^\infty \langle A(x \pm t\xi) - A(\pm t\xi), \xi\rangle\,\mathrm{d}t = \mp\int_0^\infty \langle A(x\pm t\xi), \xi\rangle\,\mathrm{d}t.$$

Note that the second equality is a consequence of the transversal gauge condition $\langle A(y), y\rangle = 0$, for all $y \in \mathbb{R}^3$. To simplify this expression we first make the change of variables $t \mapsto s$ defined by

$$s = s_0 \pm t|\xi| \qquad x = b + s_0\omega \qquad \langle b, \omega\rangle = 0 \qquad \omega = \frac{\xi}{|\xi|}$$

which leads to the equation

$$\Phi_\pm(x,\xi) = \int_{\pm\infty}^{s_0} \langle A(b + s\omega), \omega\rangle\,\mathrm{d}s. \tag{2.3}$$

Then, we rewrite this integral into spherical coordinates (see figure 1). Let

$$x(s) = b + s\omega \qquad u(s) = \sqrt{(b_1 + s\omega_1)^2 + (b_2 + s\omega_2)^2}$$

and $\varphi(s)$ be the colatitude of $x(s)$ (defined by (1.3)). Since

$$\sin(\varphi(s)) = u(s)/|x(s)| \qquad \cos(\varphi(s)) = (b_3 + s\omega_3)/|x(s)|$$

and taking into account that

$$|x(s)|^2 = |b|^2 + s^2 \qquad |\omega|^2 = 1 \qquad \langle x(s), \omega\rangle = s \qquad \frac{\mathrm{d}}{\mathrm{d}s}\cos(\varphi(s)) = -\sin(\varphi(s))\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s}$$

we get

$$\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} = \frac{sb_3 - \omega_3|b|^2}{|x(s)|^2 u(s)}.$$

On the other hand

$$\langle \mathbf{e}_{\varphi(s)}, \omega\rangle = \frac{1}{|x(s)|u(s)}(x_3(s)\langle x(s), \omega\rangle - \langle (0, 0, |x(s)|^2), \omega\rangle) \\ = \frac{|x(s)|^2\omega_3 - sx_3(s)}{|x(s)|u(s)} = \frac{sb_3 - |b|^2\omega_3}{|x(s)|u(s)}$$

which leads to

$$\langle A(x(s)), \omega\rangle = \frac{a(\varphi(s))}{|x(s)|}\langle \mathbf{e}_{\varphi(s)}, \omega\rangle \\ = \frac{a(\varphi(s))}{|x(s)|}\frac{sb_3 - |b|^2\omega_3}{|x(s)|u(s)} = a(\varphi(s))\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s}.$$

Thus, we can make the change of variables $s \mapsto \varphi(s)$ in (2.3) and, since $\varphi(s_0) = \varphi_x$ and $\varphi(\pm\infty) = \varphi_{\pm\omega} = \varphi_{\pm\xi}$, we get

$$\Phi_\pm(x,\xi) = \int_{\varphi_{\pm\xi}}^{\varphi_x} a(\varphi)\,\mathrm{d}\varphi \qquad |x| \geqslant R > 0 \tag{2.4}$$

in particular, for $|x| \geqslant R > 0$,

$$\nabla_x \Phi_\pm(x,\xi) = \frac{1}{|x|}\partial_{\varphi_x}\Phi_\pm(x,\xi)\mathbf{e}_{\varphi_x} = A(x). \tag{2.5}$$

The stationary scheme developed below makes intensive use of symbolic calculus (see [Tay81]), so we have to fix some notation on PDO. In the following we call $\mathcal{S}^m(\mu)$ the set of functions $p \in C^\infty(\mathbb{R}^6)$ satisfying, for all multi-indices α and β, the estimates

$$\left|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)\right| \leqslant C_{\alpha,\beta}\langle x\rangle^{m-|\alpha|}\langle\xi\rangle^{\mu-|\beta|}$$

and $\mathcal{S}^m = \cap_{\mu\in\mathbb{Z}}\mathcal{S}^m(\mu)$. We set $P = \mathrm{Op}\,(p(x,\xi)) = p(x,D)$ for the PDO with symbol $p \in \mathcal{S}^m$ defined for $u \in \mathcal{S}(\mathbb{R}^3)$ by

$$(Pu)(x) = \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi)\frac{\mathrm{d}\xi}{(2\pi)^{3/2}}$$

where $\hat{u}$ denotes the Fourier transform of u

$$\hat{u}(\xi) = \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle} u(x)\frac{\mathrm{d}x}{(2\pi)^{3/2}}. \tag{2.6}$$

With this notation an operator with symbol $p \in \mathcal{S}^m$ is bounded (compact) if $m \leqslant 0$ ($m < 0$). Now we are able to define the identifications $J_\pm$.

Lemma 2.1. *Let us fix* $\lambda \in (0,+\infty)$, $r \in (0,\lambda/2)$, *and let* $\psi, \eta \in C^\infty(\mathbb{R}^3,[0,1])$ *be cut-off functions satisfying:*

(i) $\eta(x) = 0$ *if* $|x| \leqslant R$ *and* $\eta(x) = 1$ *if* $|x| \geqslant R+1$
(ii) $\psi(\xi) = 1$ *if* $||\xi| - \lambda| \leqslant r$ *and* $\psi(\xi) = 0$ *if* $||\xi| - \lambda| \geqslant 2r$.

For any choice of functions η *and* ψ *we set* $J_\pm = \mathrm{Op}\,(j_\pm(x,\xi))$ *where*

$$j_\pm(x,\xi) = \mathrm{e}^{\mathrm{i}\Phi_\pm(x,\xi)}\eta(x)\psi(\xi). \tag{2.7}$$

Then $j_\pm \in \mathcal{S}^0$ *and* $J_\pm$ *is a bounded operator in* $\mathbb{L}^2(\mathbb{R}^3)$. *Additionally, the symbol of the effective perturbation* $T_\pm = HJ_\pm - J_\pm H_0 = \mathrm{Op}\,(t_\pm(x,\xi))$ *belongs to* $\mathcal{S}^m$ *for all* $m \in \mathbb{Z}$.

Proof. Since, by (2.4), $\Phi_\pm(x,\xi)$ is a homogeneous function of x and ξ (for $|x| \geqslant R$) of degree 0, it satisfies in the whole phase space the estimate

$$\left|\partial_x^\alpha \partial_\xi^\beta \Phi_\pm(x,\xi)\right| \leqslant C_{\alpha,\beta}|x|^{-|\alpha|}|\xi|^{-|\beta|} \qquad \forall\alpha,\beta \in \mathbb{N}^3 \quad |x| \geqslant R.$$

Thus taking into account the definition of cut-off functions η and ψ we get that $j_\pm \in \mathcal{S}^0$ and $J_\pm$ is bounded by the Calderon–Vaillancourt theorem. Now we calculate the symbol of the effective perturbation

$$t_\pm(x,\xi) = \mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle}(H - |\xi|^2)\,\mathrm{e}^{\mathrm{i}\langle x,\xi\rangle} j_\pm(x,\xi).$$

Using (2.1), (2.7), and that $\nabla_x\Phi_\pm(x,\xi) = A(x)$, by (2.5), we get

$$t_\pm(x,\xi) = \mathrm{e}^{\mathrm{i}\Phi_\pm(x,\xi)}\,(-2\mathrm{i}\langle\xi,\nabla_x\eta(x)\rangle - \Delta_x\eta(x))\,\psi(\xi).$$

Thus $t_\pm$ has a compact support in x and ξ and belongs to $\mathcal{S}^m$ for all $m \in \mathbb{Z}$. □

2.2. Existence and completeness of $W_\pm$

Our proof of the existence and asymptotic completeness of modified wave operators (1.6) is based on the scheme of smooth perturbations. Then it relies on the well-known limiting absorption principle:

Theorem 2.1 (Limiting absorption principle)**.** *Let H be the operator (1.5) with potential (1.1) and $\langle x\rangle = (1+|x|^2)^{1/2}$. Then, for all bounded interval $\Lambda \subset (0,\infty)$, disjoint from 0, the operator function $\langle x\rangle^{-s}R(z)^n\langle x\rangle^{-s}$, $s > n-1/2$, is (Hölder-) continuous in norm in the region* $\mathrm{Re}(z) \in \Lambda$, $\pm\mathrm{Im}(z) \in (0,1]$ *and*

$$\sup_{\substack{\mathrm{Re}\, z\in\Lambda \\ 1\geqslant|\mathrm{Im}\, z|>0}} \|\langle x\rangle^{-s}R(z)^n\langle x\rangle^{-s}\| \leqslant c \qquad \forall s > n-1/2. \tag{2.8}$$

In particular, the spectrum of H in Λ is absolutely continuous and the operators $\langle x\rangle^{-s}$, $s > 1/2$ are H-smooth on Λ (in the sense of Kato).

This result can easily be derived from the Mourre commutator method [Jen85] and the absence of positive eigenvalues for the operator (1.5) [IU71]. Now, since $t_\pm$ is short-range in the whole space, our proof of existence and completeness of wave operators relies only on the theorem 2.2 in contrast to [RY02b] where the radiation estimate was also needed.

Proposition 2.3. *Let E and E_0 be, respectively, the spectral measures of H and H_0, and $J_\pm$ be constructed as under the assumptions (i) and (ii) of lemma 2.1. Set $\Lambda = (\lambda - r, \lambda + r)$ then the wave operators $W_\pm(H, H_0, J_\pm)$ and $W_\pm(H_0, H, J_\pm^*)$ exist, are isometric on, respectively, $E_0(\Lambda)$ and $E(\Lambda)$ and are adjoint one to the other. Additionally, asymptotic completeness holds for the triple $(H, H_0, J_\pm)$, that is*

$$\mathrm{Ran}(W_\pm(H, H_0, J_\pm)E_0(\Lambda)) = \mathrm{Ran}(E(\Lambda))$$
$$\mathrm{Ran}(W_\pm(H_0, H, J_\pm^*)E(\Lambda)) = \mathrm{Ran}(E_0(\Lambda)).$$

Proof. Since the operators $\langle x\rangle^{-s}$ are H and H_0-smooth for all $s > 1/2$, the effective perturbation admits a decomposition into a product of smooth perturbations

$$T_\pm = \langle x\rangle^{-1}(\langle x\rangle T_\pm\langle x\rangle)\langle x\rangle^{-1} \tag{2.9}$$

because the PDO $\langle x\rangle T_\pm\langle x\rangle$ belongs to $\mathcal{S}^m$ for all $m \in \mathbb{Z}$ and so is a bounded operator. This is sufficient to prove the existence of $W_\pm(H, H_0, J_\pm)$ and $W_\pm(H_0, H, J_\pm^*)$ which are obviously adjoint one to the other. Now, by the chain rule, isometricity and completeness of $W_\pm(H, H_0, J_\pm)$ are, respectively, equivalent to

$$W_\pm(H_0, H, J_\pm^*)W_\pm(H, H_0, J_\pm)E_0(\Lambda) = W_\pm(H_0, H_0, J_\pm^*J_\pm)E_0(\Lambda) = E_0(\Lambda)$$
$$W_\pm(H, H_0, J_\pm)W_\pm(H_0, H, J_\pm^*)E(\Lambda) = W_\pm(H, H, J_\pm J_\pm^*)E(\Lambda) = E(\Lambda).$$

The operator $J_\pm^*J_\pm - \psi^2(D)$ is compact since its principal symbol, equal to $(\eta(x)-1)\psi^2(\xi)$, is compactly supported in x. Together with the identity $\psi^2(D)E_0(\Lambda) = E_0(\Lambda)$ this leads to

$$\begin{aligned} W_\pm(H_0, H_0, J_\pm^*J_\pm)E_0(\Lambda) &= W_\pm(H_0, H_0, \psi^2(D)E_0(\Lambda)) \\ &= W_\pm(H_0, H_0, E_0(\Lambda)) = E_0(\Lambda). \end{aligned}$$

Then, $W_\pm(H, H_0, J_\pm)$ are isometric. Asymptotic completeness goes on the same way remarking also that $E(\Lambda) - E_0(\Lambda)$ is compact. □

Finally let us check that the wave operators constructed here coincide with the usual ones constructed in [LT87].

Proposition 2.4. *Under assumptions* (i) *and* (ii) *of lemma 2.1 we have*

$$W_{\pm}(H, H_0, J_{\pm}) = W_{\pm}(H, H_0, Id)\psi(D)$$

for any choice of functions η *and* ψ.

Proof. The proof relies on the stationary phase formula applied to the integral

$$(J_{\pm}\,\mathrm{e}^{-\mathrm{i}tH_0}u)(x) = \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle - \mathrm{i}t|\xi|^2} j_{\pm}(x,\xi)\hat{u}(\xi)\frac{\mathrm{d}\xi}{(2\pi)^{3/2}}.$$

Since the stationary points $\xi_0 = \xi_0(t)$ of this integral are $\xi_0 = x/(2t)$ we get, for $u \in \mathcal{S}(\mathbb{R}^3)$, the asymptotics

$$(J_{\pm}\,\mathrm{e}^{-\mathrm{i}tH_0}u)(x) = \frac{\mathrm{e}^{\mp\mathrm{i}d\pi/4}}{(2t)^{d/2}}\,\mathrm{e}^{\mathrm{i}|x|^2/(4t)+\mathrm{i}\Phi_{\pm}(x,x/(2t))}\hat{u}(x/(2t))\eta(x)\psi(x/(2t)) + \mathrm{r}_{\pm}(x,t)$$

where $\mathrm{r}_{\pm}(x,t)$ tends to 0 in $\mathbb{L}^2(\mathbb{R}^3)$ as $t \to \pm\infty$. Now, by (2.4), we have $\Phi_{\pm}(x, x/(2t)) = 0$ for $|x| \geqslant R$ so the phase factor $\exp(\mathrm{i}\Phi_{\pm}(x, x/(2t)))$ is inessential and

$$\lim_{t\to\pm\infty}(J_{\pm}\,\mathrm{e}^{-\mathrm{i}tH_0} - \mathrm{e}^{-\mathrm{i}tH_0}\psi(D))u = \lim_{t\to\pm\infty}(\eta - 1)\,\mathrm{e}^{-\mathrm{i}tH_0}\psi(D)u = 0$$

since $\eta - 1$ is H_0-compact. In conclusion, the usual wave operators exist and coincide with the wave operators of proposition 2.3. □

Remark 2.5. If a 'short-range' electromagnetic perturbation (V_0, A_0)

$$|V_0(x)| + |A_0(x)| + |\mathrm{div}A_0(x)| \leqslant C\langle x\rangle^{-\rho} \qquad \rho > 1 \tag{2.10}$$

is added to the operator H, then all the results of this section remain true without changing the definition (2.7) of identifications $J_{\pm}$. The additional terms $\tilde{T}_{\pm}$ arising in the effective perturbation $T_{\pm}$ are short-range. Since $\rho > 1$, and by theorem 2.2, they admit a factorization into a product of H-smooth operators similar to (2.9)

$$\tilde{T}_{\pm} = \langle x\rangle^{-\rho/2}(\langle x\rangle^{\rho/2}\tilde{T}_{\pm}\langle x\rangle^{\rho/2})\langle x\rangle^{-\rho/2}.$$

3. The scattering matrix

In this section we consider the SM for the pair H, H_0 and its stationary representation. We do not give a proof of formula (1.7) (a complete justification can be found in [Yaf00]), but we rewrite it into a sum of bounded operators on $\mathbb{L}^2(\mathbb{S}^2)$ which gives its precise meaning to the formula (1.7). Thus we can make the analysis of spectral properties and singularities of $S(\lambda)$ for all $\lambda > 0$.

Let us decompose formula (1.7) as follows

$$S(\lambda) = \mathcal{W}(\lambda) + S_1(\lambda) + S_2(\lambda) \tag{3.1}$$

$$S_1(\lambda) = -2\mathrm{i}\pi\Gamma_0(\lambda)J_+^*T_-\Gamma_0^*(\lambda) \tag{3.2}$$

$$S_2(\lambda) = 2\mathrm{i}\pi\Gamma_0(\lambda)T_+^*R(\lambda + \mathrm{i}0)T_-\Gamma_0^*(\lambda) \tag{3.3}$$

with $\mathcal{W}(\lambda)$ and $\Gamma_0(\lambda)$ given by (1.8) and (1.9). In the following three propositions we analyse separately the terms $\mathcal{W}(\lambda)$, $S_1(\lambda)$ and $S_2(\lambda)$.

Proposition 3.1. *The operator* $\mathcal{W}(\lambda)$ *defined by (1.8) is the operator of multiplication by the function* $\mathrm{w}(\omega)$ *defined on* $\mathbb{S}^2$ *by (1.10).*

Proof. First remark that the commutator

$$[H_0, J_+^* J_-] = T_+^* J_- + J_+^* T_-$$

admits a factorization into a sum of products of H_0-smooth operators. Then the wave operator $W_+(H_0, H_0, J_+^* J_-)$ is well defined, commutes with H_0 (by the interwinning property), and so it reduces to multiplication by the operator-valued function $\mathcal{W}(\lambda)$ in the spectral representation of H_0. Up to compact terms the operator $J_+^* J_-$ is the PDO with principal symbol $\exp(\mathrm{i}\Theta(x, \xi))$ with

$$\Theta(x, \xi) = \Phi_-(x, \xi) - \Phi_+(x, \xi)$$

taking into account (2.4) we obtain that Θ does not depend on x

$$\Theta(x, \xi) = \int_{\varphi_{-\xi}}^{\varphi_x} a(\varphi)\,\mathrm{d}\varphi - \int_{\varphi_\xi}^{\varphi_x} a(\varphi)\,\mathrm{d}\varphi = \int_{\varphi_{-\xi}}^{\varphi_\xi} a(\varphi)\,\mathrm{d}\varphi =: \Theta(\xi). \tag{3.4}$$

Now since the operator $\exp(\mathrm{i}\Theta(D))$ commutes with H_0 we get

$$\begin{aligned} W_+(H_0, H_0, J_+^* J_-)E_0(\Lambda) &= s - \lim_{t\to\pm\infty} \mathrm{e}^{-\mathrm{i}tH_0} J_+^* J_-\, \mathrm{e}^{-\mathrm{i}tH_0} E_0(\Lambda) \\ &= s - \lim_{t\to\pm\infty} \mathrm{e}^{-\mathrm{i}tH_0}\, \mathrm{e}^{\mathrm{i}\Theta(D)}\, \mathrm{e}^{-\mathrm{i}tH_0} E_0(\Lambda) \\ &= \mathrm{e}^{\mathrm{i}\Theta(D)} E_0(\Lambda). \end{aligned}$$

The function Θ is obviously homogeneous of degree 0, by (3.4). Together with the obvious identity $\varphi_{-\omega} = \pi - \varphi_\omega$ this leads to $\exp(\mathrm{i}\Theta(\sqrt{\lambda}\omega)) = \exp(\mathrm{i}\Theta(\omega)) = \mathrm{w}(\omega)$. Then in the spectral representation where H_0 is diagonal the operator $\mathcal{W}(\lambda)$ reduces to the operator of multiplication by the function (1.10). □

Remark 3.2. From the physical point of view, $\Theta(x, \xi)$ is the circulation of the magnetic potential $A(x)$ along the 'closed' contour symbolized by dotted lines in figure 1. In particular, the calculation of function Θ is independent of the gauge chosen for $A(x)$. Thus the scheme developed before applies to any magnetic potential $\tilde{A}(x)$ satisfying $\mathrm{curl}\,(\tilde{A}) = \mathrm{curl}\,(A)$; however, the usual wave operators $W_\pm(\tilde{H}, H_0)$, with $\tilde{H} = (D - \tilde{A})^2$, should not exist if the transversal gauge is not assumed.

Here we note that the kernel of $\mathcal{W}(\lambda)$ is $\mathrm{w}(\omega)\delta(\omega, \omega')$ where δ denotes the Dirac distribution on $\mathbb{S}^2$. Below we show that the kernel of $S(\lambda)$ does not contain any other singularity.

Proposition 3.3. *The operator $S_1(\lambda)$, defined in (3.2), is an integral operator on $\mathbb{S}^2$ with a smooth kernel $\mathrm{s}_1(\omega, \omega'; \lambda) \in C^\infty(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}_+^*)$. In particular, $S_1(\lambda)$ belongs to the Hilbert–Schmidt class.*

Proof. By equations (1.9), (2.6), the operator $S_1(\lambda) = -2\mathrm{i}\pi\,\Gamma_0(\lambda) J_+^* T_- \Gamma_0^*(\lambda)$ is the restriction of a PDO on $\mathbb{L}^2(\mathbb{R}_\xi^3)$ with amplitude $\overline{j_+(x, \xi)} t_-(x, \xi')$ to the sphere $|\xi|^2 = |\xi'|^2 = \lambda$, thus it is an integral operator on $\mathbb{L}^2(\mathbb{S}^2)$ with the kernel

$$\mathrm{s}_1(\omega, \omega'; \lambda) = -\frac{\mathrm{i}\sqrt{\lambda}}{8\pi^2} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\sqrt{\lambda}\langle\omega' - \omega, x\rangle}\, \overline{j_+(x, \sqrt{\lambda}\omega)} t_-(x, \sqrt{\lambda}\omega')\,\mathrm{d}x. \tag{3.5}$$

Since the amplitude $\overline{j_+(x, \sqrt{\lambda}\omega)} t_-(x, \sqrt{\lambda}\omega')$ is compactly supported in x (due to the presence of derivatives of function η defined in lemma 2.1) the integral above obviously converges.

Differentiating expression (3.5) we get that $s_1(\omega, \omega'; \lambda)$ is a C^∞-function. In particular $|s_1(\omega, \omega'; \lambda)|^2$ is bounded and the Hilbert–Schmidt norm

$$\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} |s_1(\omega, \omega'; \lambda)|^2 \, d\omega \, d\omega'$$

of $S_1(\lambda)$ is finite. □

Proposition 3.4. *The operator $S_2(\lambda)$, defined in (3.3), is an integral operator on $\mathbb{S}^2$ with a smooth kernel $s_2(\omega, \omega'; \lambda) \in C^\infty(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}_+^*)$. $S_2(\lambda)$ belongs to the Hilbert–Schmidt class.*

Proof. Let $\psi_0(x, \xi) = \exp(\mathrm{i}\langle \xi, x\rangle)$, then the kernel of the operator $S_2(\lambda) = 2\mathrm{i}\pi \Gamma_0(\lambda) T_+^* R(\lambda + \mathrm{i}0) T_- \Gamma_0^*(\lambda)$ is formally defined by the expression

$$s_2(\omega, \omega'; \lambda) = \frac{\mathrm{i}\sqrt{\lambda}}{8\pi^2} (T_+^* R(\lambda + \mathrm{i}0) T_- \psi_0(\cdot, \sqrt{\lambda}\omega'), \psi_0(\cdot, \sqrt{\lambda}\omega))_{\mathbb{L}^2(\mathbb{R}^3)}. \tag{3.6}$$

Formula (3.6) is automatically justified if its right-hand side is a continuous function of ω, ω', λ. The derivatives $\partial_\omega^\alpha \partial_{\omega'}^{\alpha'} \partial_\lambda^m s_2(\omega, \omega'; \lambda)$ are given by a sum of terms of the form

$$\begin{aligned}(T_+^* R^n(\lambda + \mathrm{i}0) T_- \langle x\rangle^{\beta'} \psi_0(\cdot, \sqrt{\lambda}\omega'), \langle x\rangle^{\beta} \psi_0(\cdot, \sqrt{\lambda}\omega))_{\mathbb{L}^2(\mathbb{R}^3)} \\ = (\langle x\rangle^{-n} R^n(\lambda + \mathrm{i}0) \langle x\rangle^{-n} Q_- \langle x\rangle^{-2} \psi_0(\cdot, \sqrt{\lambda}\omega'), Q_+ \langle x\rangle^{-2} \psi_0(\cdot, \sqrt{\lambda}\omega))_{\mathbb{L}^2(\mathbb{R}^3)}\end{aligned}$$

with $Q_+ = \langle x\rangle^n T_+ \langle x\rangle^{\beta+2}$, $Q_- = \langle x\rangle^n T_- \langle x\rangle^{\beta'+2} \in \mathcal{S}^m$ for all $m \in \mathbb{Z}$ and $|\alpha| + |\alpha'| + m = |\beta| + |\beta'| + n$. Since the operators $Q_\pm$ and $\langle x\rangle^{-n} R^n(\lambda + \mathrm{i}0) \langle x\rangle^{-n}$ are bounded on $\mathbb{L}^2(\mathbb{R}^3)$ (by theorem 2.2) and taking into account that $\langle x\rangle^{-2} \psi_0(\cdot, \xi) \in \mathbb{L}^2(\mathbb{R}^3)$, those expressions are correctly defined and bounded. Finally, since $\psi_0(\cdot, \sqrt{\lambda}\omega)$, $\langle x\rangle^{-n} R^n(\lambda + \mathrm{i}0) \langle x\rangle^{-n}$ are continuous in λ and ω, we have shown that $s_2(\omega, \omega'; \lambda)$ is a C^∞-function. In particular $|s_2(\omega, \omega'; \lambda)|^2$ is bounded and the Hilbert–Schmidt norm of $S_2(\lambda)$ is finite. □

Combining propositions 3.1, 3.3 and 3.4 we obtain

Theorem 3.5. *Let H be the operator (1.5) with potential (1.1), $S(\lambda)$ be the SM for the pair $H, H_0 = -\Delta$ and $\mathcal{W}(\lambda)$ be the operator of multiplication on $\mathbb{S}^2$ by the function* w *defined in (1.10). Then the operator $S(\lambda) - \mathcal{W}(\lambda)$ has an infinitely-smooth kernel, in particular it belongs to the Hilbert–Schmidt class.*

We can now prove the two essential results on spectral properties of the SM for the pair H, H_0.

Theorem 3.6. *Let H be the operator (1.5) with potential (1.1) and $S(\lambda)$ be the SM for the pair $H, H_0 = -\Delta$, then the essential spectrum of $S(\lambda)$ is given by (1.11).*

Proof. Since $\mathcal{W}(\lambda)$ is the operator of multiplication by w, its (continuous) spectrum coincides with the range of the function w. Since the function a in (1.1) is positive and taking into account relation (1.4) the range of the function (3.4) equals the interval $[-\phi, \phi]$ and the spectrum of $\mathcal{W}(\lambda)$ is the image of this interval by the function $\upsilon \mapsto \exp(\mathrm{i}\upsilon)$. Finally, since $S(\lambda) - \mathcal{W}(\lambda)$ is Hilbert–Schmidt, and also compact, thanks to Weyl theorem the essential spectrum of $S(\lambda)$ coincides with the essential spectrum of $\mathcal{W}(\lambda)$ that is (1.11). □

Theorem 3.7. *Let H be the operator (1.5) with potential (1.1) and $S(\lambda)$ be the SM for the pair $H, H_0 = -\Delta$, then the total scattering cross-section $\Sigma_{\mathrm{tot}}(\omega_0; \lambda)$ defined by (1.12) and (1.13)) is finite for any incident direction ω_0.*

Proof. The kernel of the principal part $\mathcal{W}(\lambda)$ is $\mathrm{w}(\omega)\delta(\omega, \omega')$ where δ denotes the Dirac distribution on $\mathbb{S}^2$. In particular, its support is concentrated on the diagonal $\omega = \omega'$. Off the, diagonal the kernel of $S(\lambda)$ reduces to the sum $\mathrm{s}_1(\omega, \omega'; \lambda) + \mathrm{s}_2(\omega, \omega'; \lambda)$. Since s_1 and s_2 are infinitly smooth functions the integral (1.13) converges and the total scattering cross-section is finite for all ω_0. □

Let us make some comments on the nature of the spectrum of the operators $\mathcal{W}(\lambda)$ and $S(\lambda)$. Since $a \in C_0^\infty(0, \pi)$ the function w is constant in a conical neighbourhood of the x_3 axis (see figure 1); thus any function $u \in \mathbb{L}^2(\mathbb{S}^{d-1})$ supported in a small enough neighbourhood of the point $(0, 0, 1)$ (respectively $(0, 0, -1)$) is an eigenfunction of the operator $\mathcal{W}(\lambda)$ associated with the eigenvalue $\exp(-\mathrm{i}\phi)$ (respectively $\exp(\mathrm{i}\phi)$). Consequently, the spectrum of the operator $\mathcal{W}(\lambda)$ is absolutely continuous on the arc $[\exp(\mathrm{i}\phi), \exp(-\mathrm{i}\phi)]$ (contained in the unit circle) except at the points $\exp(\pm\mathrm{i}\phi)$ which are eigenvalues of infinite multiplicity. Note that if $\phi = n\pi$, $n \in \mathbb{N}^*$, then the spectrum of $\mathcal{W}(\lambda)$ covers the unit circle and is absolutely continuous except for the eigenvalue $(-1)^n$. Now, considering $S(\lambda)$ as a compact perturbation of $\mathcal{W}(\lambda)$, the eigenvalues $\exp(\pm\mathrm{i}\phi)$ would split into a discrete set of eigenvalues accumulating at the points $\exp(\pm\mathrm{i}\phi)$ (possibly equal). We can also note that if $\phi \geqslant \pi$ then the spectrum of $S(\lambda)$ covers the unit circle.

Remark 3.8. The result of theorem 3.7 is preserved under short-range perturbations (V_0, A_0) if we suppose that (2.10) is satisfied for some $\rho > 3$.

4. The Aharonov–Bohm effect in dimension 3

Finally, we want to make some remarks on the Aharonov–Bohm effect. Since the three-dimensional example of magnetic field treated here is compactly supported it is somewhat natural to compare our results with those obtained in the two-dimensional case. Let the Aharonov–Bohm Hamiltonian be the operator $H_{AB} = (D - A_{AB}(x))^2$, on $\mathbb{L}^2(\mathbb{R}^2)$, with the magnetic potential

$$A_{AB}(x) = a(\theta_x)\frac{(-x_2, x_1)}{|x|^2} \qquad |x| \geqslant R > 0 \tag{4.1}$$

where $a \in C^\infty(\mathbb{R})$ is a 2π-periodic function of the polar angle θ_x associated with $x = (x_1, x_2)$. The family of potentials satisfying (4.1) includes all compactly supported magnetic fields in dimension 2. For potential (4.1) an analysis, similar to that made here, was developed in [RY02a]. With the notation

$$\phi_{AB} = \int_0^{2\pi} a(\vartheta)\,\mathrm{d}\vartheta \qquad f(\theta) = \int_\theta^{\theta+\pi} a(\vartheta)\,\mathrm{d}\vartheta$$

it is shown that for $S_{AB}(\lambda)$, the SM associated with the pair H_{AB}, H_0, we have that

$$\sigma_{\mathrm{ess}}(S_{AB}(\lambda)) = \exp(\mathrm{i}f(\mathbb{R})) \cup \exp(-\mathrm{i}f(\mathbb{R})) \tag{4.2}$$

and the differential scattering cross-section admits the asymptotic

$$\Sigma_{\mathrm{diff}}(\omega, \omega_0; \lambda) = \frac{1}{2\pi\sqrt{\lambda}}\frac{\sin^2(\phi_{AB}/2)}{\sin^2(\theta/2)} + \mathcal{O}\left(\frac{\ln(\theta)}{\theta}\right) \tag{4.3}$$

as $\omega \to \omega_0$, with $|\omega - \omega_0| = 2\sin(\theta/2)$. Equations (4.2) and (4.3) generalize the results obtained by various authors (see [Rui83]) for the radial potential (4.1) with a constant function $a = \phi_{AB}/(2\pi)$:

$$\sigma(S_{AB}(\lambda)) = \sigma_{pp}(S_{AB}(\lambda)) = \{\mathrm{e}^{\mathrm{i}\phi_{AB}/2}, \mathrm{e}^{-\mathrm{i}\phi_{AB}/2}\}$$

and

$$\Sigma_{\mathrm{diff}}^{AB}(\omega, \omega_0; \lambda) = \frac{1}{2\pi\sqrt{\lambda}}\frac{\sin^2(\phi_{AB}/2)}{\sin^2(\theta/2)}.$$

If we compare (1.11) with (4.2) we remark that both SM have intermediary spectral properties between the general cases of short and long-range potentials where, respectively, the essential spectrum reduces to $\{1\}$ or covers the whole unit circle (see [RY02b]). From (4.3) we see that in dimension 2 the total scattering cross-section is infinite except if the magnetic flux $\phi_{AB} \in 2\pi\mathbb{Z}$, in contrast, for potential (1.1), in dimension 3, this situation does not appear.

In contrast to the two-dimensional case, only a few authors have been interested in the three-dimensional case [Tam95, BR00], so the potential (1.1) can be regarded as an interesting example. Since the family of potentials satisfying (1.1) does not contain all compactly supported magnetic fields in dimension 3 we could not exclude the existence of such fields with infinite total cross-section, but it seems that the situation described in this paper is very general. Indeed, let us consider a magnetic potential A obtained from an arbitrary compactly supported magnetic field, that is

$$\operatorname{curl} A(x) = 0 \Longleftrightarrow A(x) = \nabla\Phi(x) \tag{4.4}$$

for large $|x|$ and some regular function Φ. It is quite plausible that we can choose $\Phi = \Phi_\pm$, the solutions of the eikonal equation (2.2), as in the case of potential (1.1). If this conjecture is verified then we can generalize the scheme developed here to all potentials satisfying (4.4). As in section 2 we can define the wave operators of lemma 2.1 which coincide with the usual ones and thus are complete. Similarly the results of section 3 would be generalized. Remark that the results of propositions 3.3 and 3.4 do not depend on the phases $\Phi_\pm$ and so they hold for arbitrary potential A. Thus all the singularities of the SM are contained in the term $\mathcal{W}(\lambda)$ defined by (1.8). As shown in the proof of proposition 3.1, the singularities of $\mathcal{W}(\lambda)$ reduce to the Dirac singularity if the function $\Theta(x,\xi) = \Phi_-(x,\xi) - \Phi_+(x,\xi)$ is independent of x (for large $|x|$). This fact follows from the initial conjecture since

$$\nabla_x\Theta(x,\xi) = \nabla_x\Phi_-(x,\xi) - \nabla_x\Phi_+(x,\xi) = A(x) - A(x) = 0.$$

Then for any compactly supported magnetic field the SM will reduce to multiplication by the function $\exp(\mathrm{i}\Theta(\omega))$, up to a C^∞-kernel operator. So in contrast to the two-dimensional case, the total scattering cross section will always be finite in dimension 3. A final argument for this conjecture can be found in [Yaf02]. In this paper Yafaev has shown a result similar to the one conjectured here, but for short-range magnetic potentials, that the high-energy limit of the SM is the operator of multiplication by $\exp\left(\mathrm{i}\int_{\mathbb{R}}\langle A(t\omega),\omega\rangle\,\mathrm{d}t\right)$.

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